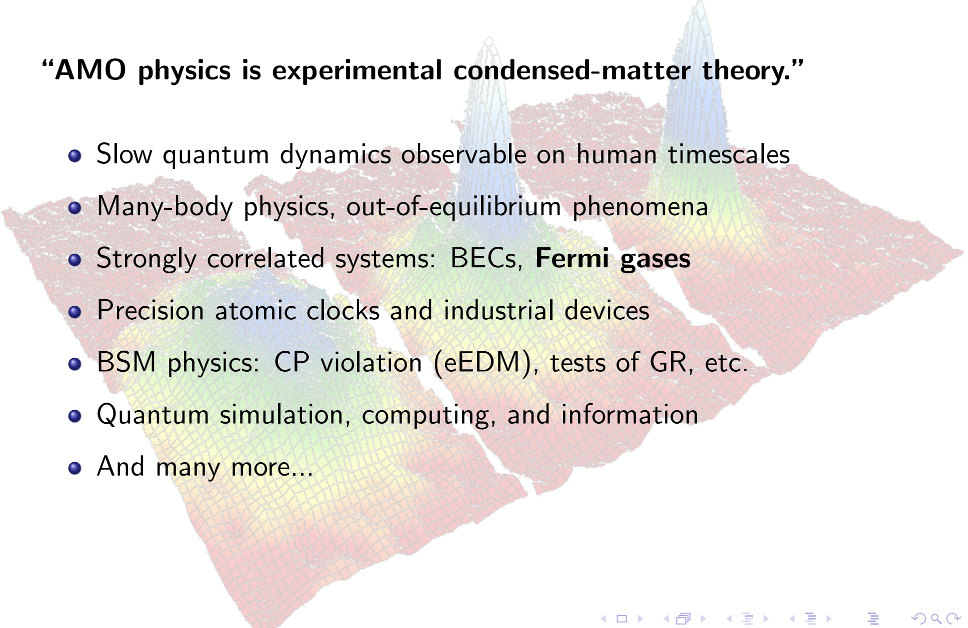
The Unitary Fermi Gas

David Grabovsky and Evan Wickenden

April 23, 2019

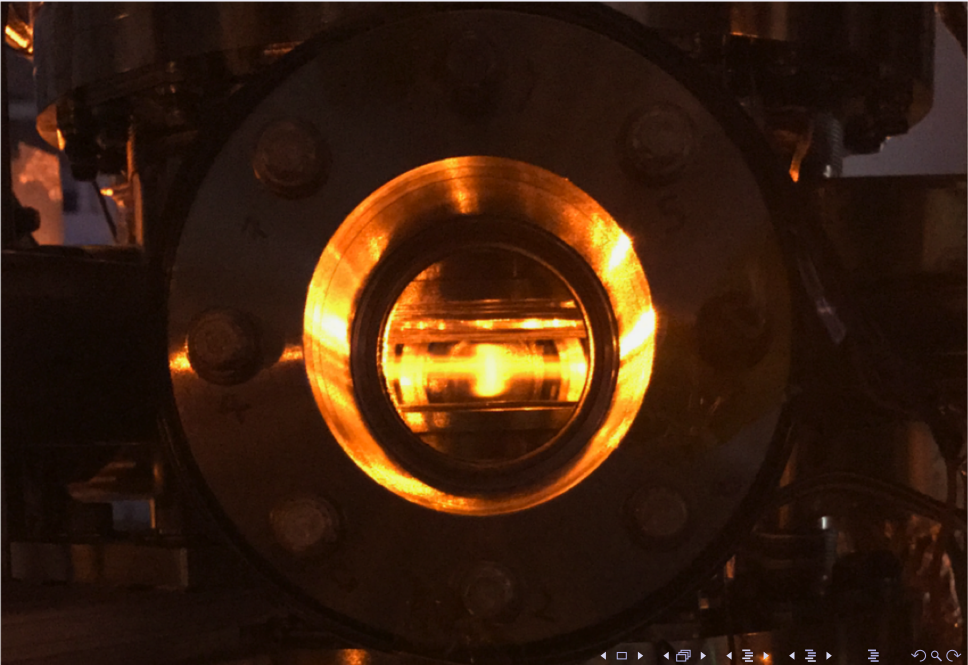
Motivation: Why Go Ultracold?



“AMO physics is experimental condensed-matter theory.”

- Slow quantum dynamics observable on human timescales
- Many-body physics, out-of-equilibrium phenomena
- Strongly correlated systems: BECs, **Fermi gases**
- Precision atomic clocks and industrial devices
- BSM physics: CP violation (eEDM), tests of GR, etc.
- Quantum simulation, computing, and information
- And many more...

Na Atoms in Sebastian Will's Lab



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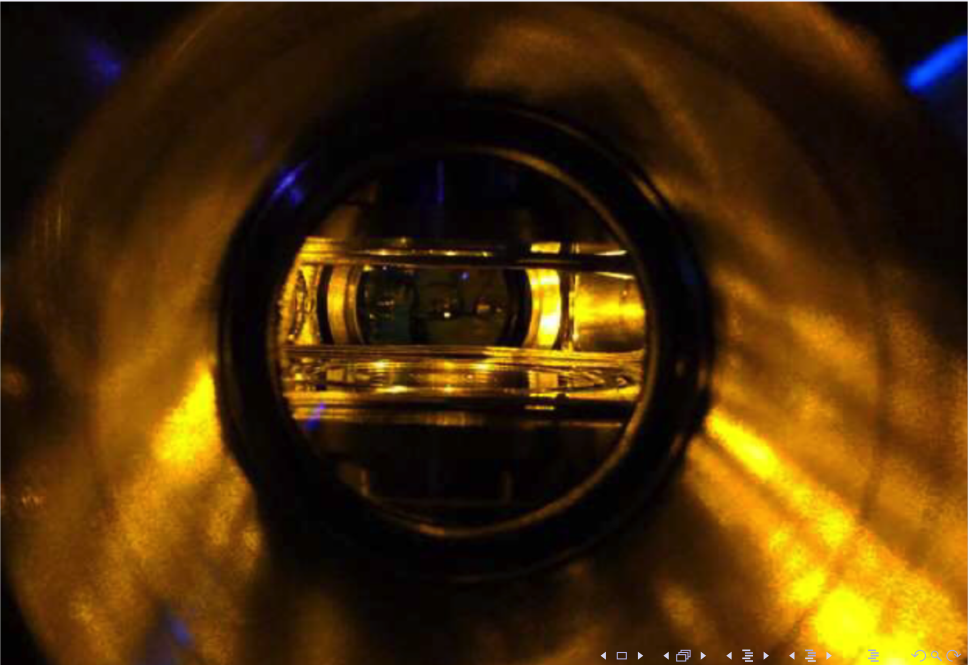

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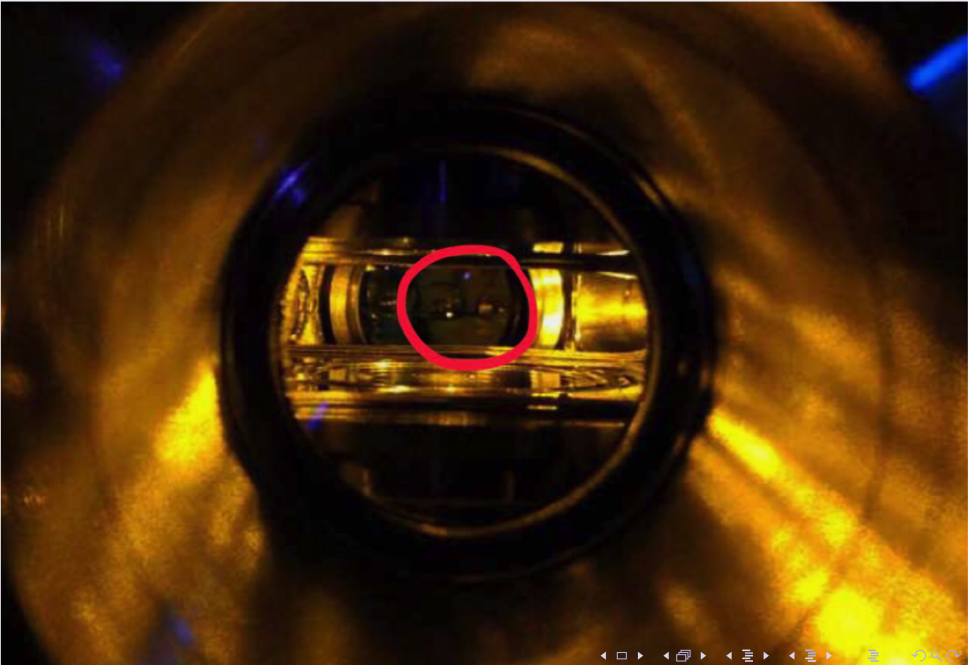


Fermi Gases: Background

1995: Cornell, Wiemann, Ketterle synthesize the first BEC (Nobel 2001).

2004-2005: The BCS-BEC crossover is experimentally observed, with unitarity achieved at the phase transition in the middle of the crossover.

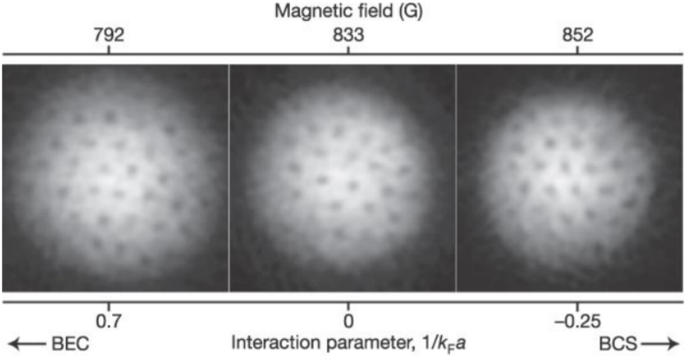


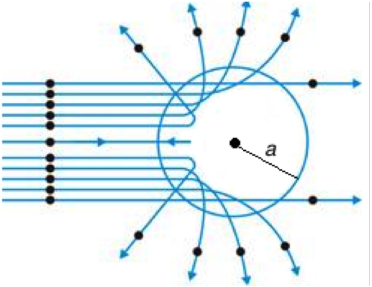
Figure: Quantized vortices in a rotating Fermi gas, demonstrating superfluidity.

The Scattering Length

Definition (scattering length)

The **scattering length** a is given by the low-energy limit

$$\lim_{k\to 0} k\cot\delta(k) = -\frac{1}{a}. \qquad (1)$$



a gives the length scale of correlations in the gas. Contact scattering against a sphere of radius a has the same cross section: $\lim_{k\to 0}\sigma = 4\pi a^2$.

A dilute, homogeneous Fermi gas only has length scales $1/k_F$ and a, so the system is fully characterized by the dimensionless parameter $k_F a$.

The **unitary Fermi gas** is the limit $k_F a \to \infty$.

Implications of Unitarity

- Infinite interaction range makes every atom coupled to every other, producing a maximally many-body state.
- The gas is self-similar at every length scale: zooming out does not decouple the system or make interactions fade.
- Self-similarity and scale invariance often accompany a phase transition: in this case, the BCS-BEC crossover.
- This lets us view unitary fermions as a strongly coupled CFT...

SURPRISE!

The Gauge/Gravity Duality

Duality is emerging as an important theme in modern physics.

Example: classical electrodynamics in vacuum

Maxwell's equations in free space are invariant under the interchange of the electric and magnetic fields, $\mathbf{E} \to \mathbf{B}, \;\; \mathbf{B} \to -\frac{1}{c^2}\mathbf{E}$.

AdS/CFT: a theory of (quantum) gravity in a "bulk" spacetime is equivalent to a conformal field theory on its boundary.

Implications: M-theory, qark-gluon plasma, superfluids... cold atoms!

Cold Atoms and Holography

The Gravity Dual of the Schrödinger Equation

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3. Holography for Galilean CFTs
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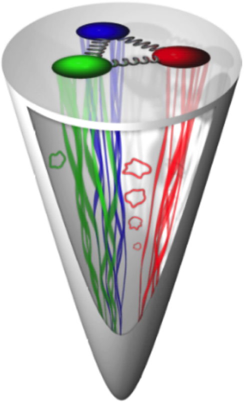

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Quantum Field Theory

QFT: quantization of classical fields by promoting them to operators

Discrete	Continuous
$$\frac{\partial L}{\partial q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)$$	$$\partial_\mu \mathcal{L} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)$$
$$[q, p] = i\hbar$$	$$[\phi(x), \pi(y)] = i\hbar\delta(x-y)$$

- Infinitely many degrees of freedom: QM is $(0+1)$-D QFT.
- Symmetries constrain the terms allowed in the Lagrangian.

Example: relativistic free scalar field

$$\mathcal{L} = \frac{1}{2}\left(\partial_t^2\phi - \nabla^2\phi\right) - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2. \qquad (2)$$

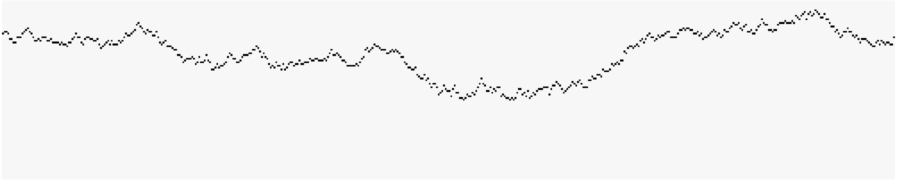
Conformal Invariance

Definition (conformal field theory)

A field theory is **scale-invariant** if a **scale transformation** $x \to \lambda x$ causes the fields to transform as $\phi \to \lambda^{-\Delta}\phi$, i.e. $\phi(x) = \lambda^{\Delta}\phi(\lambda x)$.

Example: QED and Running Couplings

In QED, the electron's "bare" charge diverges as we get closer to it (i.e. at higher energies). In CFTs, the couplings do not run.



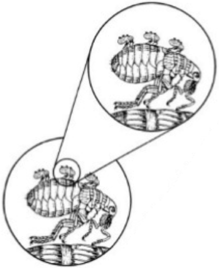
Scaling and Phase Transitions

Phase transitions $\iff$ invariance under rescalings $x \to \lambda x,\ t \to \lambda^z t$.

- For Lorentz-invariant systems, $z = 1$ since space $\equiv$ time.
- The free Schrödinger equation is invariant with $z = 2$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) = i\hbar\frac{\partial}{\partial t}\psi(x,t).$$

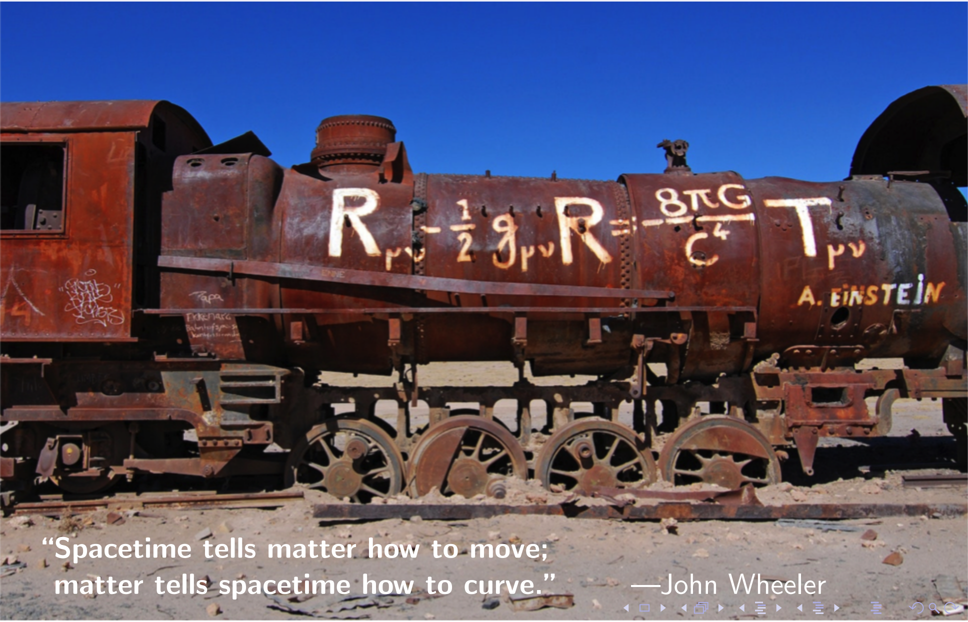
(λ^2 cancels on both sides)



"Big fleas have little fleas upon their backs to bite 'em,
And little fleas have lesser fleas, and so on, *ad infinitum*.
And great fleas, themselves, in turn, have greater fleas to go on;
While these again have greater still, and greater still, and so on.

—Augustus De Morgan

General Relativity



"Spacetime tells matter how to move; matter tells spacetime how to curve."

—John Wheeler

General Relativity

The **Einstein-Hilbert action** yields the **Einstein field equations** (EFE),

$$S = \int \left[\frac{1}{4\pi G}(R - 2\Lambda) + \mathcal{L}_{\mathrm{M}} \right] \sqrt{-g}\, \mathrm{d}^4 x; \tag{3}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{4}$$

- The **cosmological constant** Λ represents a uniform negative energy density ("vacuum energy") throughout spacetime.
- Solutions to the EFE are **metrics** $g_{\mu\nu}$, which respond to mass-energy $T_{\mu\nu}$ to determine the geometry of spacetime.

Vacuum Solutions to the EFE

$T_{\mu\nu} = 0$ implies a spacetime of constant curvature.

Euclidean signature

- $K = +1$: Sphere, S^n
- $K = 0$: Euclidean space, $\mathbb{R}^n$
- $K = -1$: Hyperbolic space, $\mathbb{H}^n$

Lorentzian signature

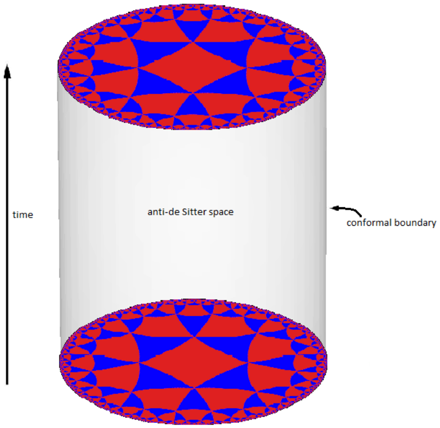
- $\Lambda > 0$: De Sitter space, dS_n
- $\Lambda = 0$: Minkowski, $\mathbb{R}^{n-1,1}$
- $\Lambda < 0$: Anti-de Sitter, AdS_n

The Lorentzian spacetimes are obtained from the Euclidean spaces by flipping the sign of the time coordinate (i.e. by Wick rotation).

String Theory on AdS

String theory is a quantum theory of gravity, and is not well understood.

In particular, it may be formulated in an AdS_n background whose **conformal boundary** is locally Minkowski in dimension $n-1$.



Holographic Duality!

- **Bekenstein-Hawking, 1974**: black hole entropy scales with area,

$$S_{\mathrm{BH}} = \frac{c^3 k_{\mathrm{B}} A}{4\hbar G} = \frac{4 k_{\mathrm{B}} A}{4\ell_{\mathrm{P}}^2}. \tag{5}$$

- **Susskind, 1995**: the boundary holographically describes the bulk.
- **Maldacena, 1997**: the Minkowski boundary $\mathbb{R}^{d-1,1}$ of AdS_{d+1} provides the spacetime for a CFT (e.g. $\mathcal{N} = 4$ SYM), whose physics is equivalent to string theory (e.g. IIB on $\mathrm{AdS}_5 \times S^5$) in the bulk.
- This is a **strong-weak duality**: the semiclassical limit of string theory (i.e. GR) corresponds to strong coupling in the CFT!

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The Plan of Attack

1. Describe the algebra of Schrödinger symmetries explicitly.
2. Embed this algebra into a higher-dimensional conformal group, making unitary fermions part of a bigger relativistic CFT.
3. Write down an AdS metric dual to this CFT, then deform it to reduce its symmetries to those of the Schrödinger equation.
4. Analyze this dual theory and develop its AdS/CFT dictionary.

The Schrödinger Algebra

The Schrödinger algebra is generated by time translations H and spatial translations P^i, rotations M^{ij}, Galilean boosts K^i, dilation D with $z = 2$, a "special conformal" transformation C, and the mass m.

Nonzero commutators of the Schrödinger algebra:

$$
\begin{aligned}
&[M^{ij}, M^{kl}] = i(\delta^{ik} M^{jl} + \delta^{jl} M^{ik} - \delta^{il} M^{jk} - \delta^{jk} M^{il}),\\
&[M^{ij}, P^k] = i(\delta^{ik} P^j - \delta^{jk} P^i), \quad [M^{ij}, K^k] = i(\delta^{ik} K^j - \delta^{jk} K^i),\\
&[D, P^i] = -iP^i, \quad [D, K^i] = iK^i, \quad [P^i, K^j] = -i\delta^{ij} m,\\
&[D, H] = 2iH, \quad [D, C] = 2iC, \quad [H, C] = iD.
\end{aligned} \tag{6}
$$

Unitary fermions are also invariant under the spin rotations $SU(2)$.

Embedding into a Conformal Group

The massless Klein-Gordon equation in $(\mathbb{R}^{d+1,1},\ \eta_{\mu\nu})$ is the relativistic wave equation, and is conformally invariant with $z = 1$:

$$\Box\psi \equiv \left(-\,\partial_0^2 + \sum_{i=1}^{d+1} \partial_i^2\right)\phi = 0. \tag{7}$$

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In light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{d+1})$,

$$\Big(- 2 \frac{\partial}{\partial x^-} \frac{\partial}{\partial x^+} + \sum_{i=1}^{d} \partial_i^2 \Big) \psi = 0. \tag{8}$$

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Make the identification $\partial/\partial x^- \equiv -im$, and let x^+ play the role of time. (Or, identify the light-cone momenta $P^\pm$ with mass and energy.)

$$\Big(2im\frac{\partial}{\partial t} + \sum_{i=1}^{d} \partial_i^2\Big)\psi = 0 \iff i\frac{\partial}{\partial t}\psi = -\frac{1}{2m}\nabla^2\psi. \tag{9}$$

The Gravitational Dual

Deform the AdS_{d+3} metric in Poincaré coordinates $(x^{\pm}, x^{i}, r)$:

$$\begin{aligned} \mathrm{d}s^2 =& \frac{1}{r^2}\Big(\eta_{\mu\nu}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + \mathrm{d}r^2\Big) = \frac{1}{r^2}\Big(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2 + \mathrm{d}r^2\Big) \longrightarrow \\ & \frac{1}{r^2}\Big(-\frac{2\,\mathrm{d}x^{+2}}{r^2} - 2\,\mathrm{d}x^{-}\mathrm{d}x^{+} + \mathrm{d}\mathbf{x}^2 + \mathrm{d}r^2\Big) = \mathrm{d}s_*^2. \end{aligned} \tag{10}$$

Generators of the conformal algebra correspond to isometries of $g_{\mu\nu}$.

Key features:

- Negative curvature $(\Lambda < 0)$ and uniform pressureless dust.
- Discrete mass spectrum $\implies$ x^{-} is compactified à la KK.

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Summary and Takeaways

Summary:

- Correlations of infinite range $a \to \infty$ in ultracold Fermi gases yield universal, scale-invariant behavior ($z = 2$).
- Viewed as an infinitely strongly coupled CFT, unitary fermions should have a dual gravitational description in the bulk.
- We achieve this by embedding the Schrödinger algebra into a conformal group, and by deforming the corresponding AdS metric to reduce its symmetries to those of the Schrödinger equation.
- The dual theory has negative cosmological constant, is dusty, and has an extra dimension. Not much else is currently understood.

Takeaway: physics is cool as hell.

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6. We thank Wikipedia for its useful discussions on scale invariance, AdS space, the Einstein-Hilbert action, and the AdS/CFT correspondence.